

# ON THREE DIMENSIONAL CONFORMALLY FLAT ALMOST COSYMPLECTIC MANIFOLDS

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ABSTRACT. In the paper there are described new examples of conformally flat three dimensional almost cosymplectic manifolds. All these manifolds form a class which was completely characterized.

## 1. INTRODUCTION

It is a difficult problem to construct explicit examples of almost cosymplectic manifolds endowed with Riemannian metrics which satisfy some classical curvature conditions like e.g. the Einstein condition, conformally flatness, local symmetry etc. Known explicit examples are of a very special structure. Many of them are some Lie groups endowed with a left invariant almost cosymplectic structures [2], [3], [5]. However with no evident connection to the mentioned above metric properties. In this view any new explicit examples are of no doubt of much worth as they may give a new impulse for the further development of the whole theory.

For an almost cosymplectic manifolds there are non-existence theorems. The oldest result of this kind is the theorem due to Z.Olszak [8], [9] which asserts that the curvature of a constant curvature almost cosymplectic manifold is zero and manifold is cosymplectic. For the almost cosymplectic manifold with Kähler leaves (see next section) holds even stronger result: in dimension  $\geq 5$  there are no conformally flat metric [4]. The case of the dimension 3 is different. We remind here the first explicit example of a conformally flat almost cosymplectic three manifold given in [4]:  $p = (x, y, z) \in U \subset \mathbb{R}^3$ ,

$$(1) \quad \begin{aligned} g &= z^2 dx^2 + \frac{e^{2ax}}{z^2} dy^2 + dz^2, \\ \xi &= \frac{\partial}{\partial z}, \quad \eta = dz, \\ \varphi \frac{\partial}{\partial x} &= \frac{z^2}{e^{ax}} \frac{\partial}{\partial y}, \quad \varphi \frac{\partial}{\partial y} = -\frac{e^{ax}}{z^2} \frac{\partial}{\partial x}. \end{aligned}$$

Remarkable the almost contact structure in this example is non-normal therefore this it is a non-cosymplectic conformally flat manifold. This is important as it is easy to construct an example of conformally flat *cosymplectic*

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three manifold. Even more to classify locally all such manifolds (cf. sect. 4).

There are no explicit examples in dimensions  $\geq 5$  and it is still the open problem whether there exist conformally flat non-flat almost cosymplectic manifolds. In [8] Z. Olszak proved that the scalar curvature  $s$  of conformally flat almost cosymplectic manifold of dimension  $\geq 5$  is non-positive which was later improved by H. Endo [6] who found the following inequality

$$\rho(X) + \rho(\varphi X) \leq -\frac{s}{n(2n-3)},$$

for the Ricci curvature  $\rho$  of arbitrary unit vector  $X \perp \xi$ .

## 2. PRELIMINARIES

An almost contact metric structure consists of four tensor fields customary denoted by  $\varphi$ ,  $\xi$ ,  $\eta$  and  $g$  where  $\varphi$  is a  $(1, 1)$  tensor field,  $\xi$  is a vector field,  $\eta$  a 1-form and  $g$  is a Riemannian metric. Moreover one requires the following relations must be satisfied

$$(2) \quad \begin{aligned} \varphi^2 &= -Id + \eta \otimes \xi, & \eta(\xi) &= 1, \\ g(\varphi X, \varphi Y) &= g(X, Y) - \eta(X)\eta(Y), \end{aligned}$$

where  $X, Y$  are arbitrary vector fields.

To each almost contact metric manifold  $M$ , i.e. manifold endowed with an almost contact metric structure, is associated a 2-form  $\Phi(X, Y) = g(\varphi X, Y)$  usually called a fundamental form. We mention that our definition of  $\Phi$  differs by sign of that given in [1] where  $\Phi(X, Y) = g(X, \varphi Y)$ . However this is explained by historical reasons.

An almost contact metric manifold  $M$  is always odd-dimensional,  $\dim M = 2n + 1$ . The fundamental form  $\Phi$  is of maximal possible rank  $2n$  as its kernel, at each point, consists of vectors  $c\xi$ ,  $c = \text{const}$ . Moreover  $\omega = \eta \wedge \Phi^n$  is a non-vanishing everywhere  $(2n + 1)$ -form hence  $M$  is orientable.

Considering the behavior of the exterior differentials of the forms  $\eta$  and  $\Phi$  we fall through different classes of almost contact metric manifolds. The more exhaustive studied class of manifolds nowadays are *contact metric* manifolds. The terminology is explained by the fact that  $\eta$  is a contact form  $\eta \wedge (d\eta)^n \neq 0$ . The basic reference to this theory is a monograph [1]. In a some sense opposite direction we have *almost cosymplectic* manifolds as they were defined by the conditions that both  $\eta$  and  $\Phi$  are closed [7].

Let  $M$  be an almost cosymplectic manifold. Then we have the following fundamental identity [8]

$$(\nabla_{\varphi X} \varphi) \varphi Y + (\nabla_X \varphi) Y - \eta(Y) \nabla_{\varphi X} \xi = 0.$$

which implies that  $\nabla_{\xi} \varphi = 0$  and  $\nabla_{\xi} \xi = 0$  hence any integral curve of  $\xi$  is a geodesic.

Let denote by  $\mathcal{F}$  a foliations of  $M$  defined by  $\mathcal{D} = \ker \eta$ . We fix a leaf  $N \in \mathcal{F}$ . A form

$$\Omega = i^* \Phi,$$

$i$  being an inclusion map, is a symplectic form on  $N$  so  $(N, \Omega)$  is a symplectic manifold. Even more  $(N, \Omega)$  can be endowed with an inherited almost Hermitian structure  $(J, G)$  in the way that  $\Omega$  becomes a fundamental form of this almost Hermitian structure. Therefore  $(N, J, G)$  may be considered as an almost Kähler manifold.

We set by definition

$$(3) \quad i_*(J\bar{X}) = \varphi i_*(\bar{X}),$$

for any tangent to  $N$  vector field. We have  $i_*(\bar{X}) = \mathcal{D}_{i(p)}$  as  $N$  is an integral submanifold of  $\mathcal{D}$ . Note

$$\varphi^2 i_*(\bar{X}) = -i_*(\bar{X}) + \eta(i_*(\bar{X}))\xi = -i_*(\bar{X}).$$

Thus  $\varphi^2|_{\mathcal{D}_p} = -Id|_{\mathcal{D}_p}$  and the linear algebra arguments imply  $\varphi(\mathcal{D}_p) = \mathcal{D}_p$ . The identity above also shows that  $J$  is an almost complex structure on  $N$ . Now for the metric  $G$  we set  $G = i^*g$ . Therefore  $(N, G)$  is a Riemannian hypersurface in  $M$ . Let  $\bar{X}, \bar{Y}$  be arbitrary tangent to  $N$  vector fields. Then

$$\begin{aligned} G(J\bar{X}, J\bar{Y}) &= g(i_*(J\bar{X}), i_*(J\bar{Y})) = g(\varphi i_*(\bar{X}), \varphi i_*(\bar{Y})) \\ &= g(i_*(\bar{X}), i_*(\bar{Y})) = G(\bar{X}, \bar{Y}), \end{aligned}$$

where we have used (3), the definition of  $G$  and (2). In similar way we can show that

$$\Omega(\bar{X}, \bar{Y}) = G(J\bar{X}, \bar{Y}).$$

Summing up all above we have

$$\begin{aligned} J^2 &= -Id, \quad G(JX, JY) = G(X, Y), \\ \Omega(X, Y) &= G(JX, Y), \quad d\Omega = 0, \end{aligned}$$

so  $(J, G)$  is an almost Kähler structure on  $N$  with  $\Omega$  as fundamental form. Of course our construction do not depend on the choice of a leaf in the sense that each leaf can be endowed with an almost Kähler structure in the way described above. However in general these structures on different leaves are different. In the case that these induced structures on any leaf are Kählerian the manifold  $M$  is said to be almost cosymplectic with Kählerian leaves [9]. Note that each three dimensional almost cosymplectic manifold clearly has Kählerian leaves as the leaves are two dimensional.

Let  $\nabla$  and  $\bar{\nabla}$  denotes the Levi-Civita connections on  $M$  and  $N$  resp. then we have the usual Gauss decomposition formula

$$\nabla_{\bar{X}} \bar{Y} = \bar{\nabla}_{\bar{X}} \bar{Y} + h(\bar{X}, \bar{Y})\bar{n},$$

where  $h$  is a second fundamental form of the Riemannian hypersurface  $N$  and  $\bar{n}$  stands for the normal vector field. The field will be determined uniquely if we choose orientation on both  $N$  and  $M$  requiring that for a given positively oriented frame  $(\bar{X}_1, \dots, \bar{X}_{2n})$  of  $T_p N$  the frame  $(\bar{n}, \bar{X}_1, \dots, \bar{X}_{2n})$  of  $T_p M$  is also positively oriented. Quite naturally we take on  $M$  an orientation given by the equivalence class of the form  $\omega = \eta \wedge \Phi^n$  and for  $N$  those one determined by the almost complex structure  $J$ . With these assumptions we

have  $\bar{n} = \xi|_N$  as  $\xi$  is unit vector field and everywhere orthogonal to any tangent space of  $N$ .

For the Weingarten operator of  $N$  we have

$$S\bar{X} = -\nabla_{\bar{X}}\bar{n} = -\nabla_{\bar{X}}\xi.$$

Again we see that formula for Weingarten operators of different leaves is exactly the same. This observation suggests to introduce a tensor field  $A$  as follows

$$AX = -\nabla_X\xi.$$

We note some properties of  $A$ . On the leaf  $N$  the tensor field  $A$  and the operator  $S$  are related by

$$(4) \quad Ai_*(\bar{X}) = i_*(S\bar{X}),$$

$A$  is symmetric  $g(AX, Y) = g(X, AY)$  and anti-commutes with  $\varphi$

$$(5) \quad \varphi A + A\varphi = 0.$$

As the vector field  $\xi$  is geodesic we have  $A\xi = -\nabla_\xi\xi = 0$  which implies  $\eta(AX) = 0$ . From (5) it follows that

$$A\varphi X = -\lambda\varphi X \quad \text{if} \quad AX = \lambda X.$$

Therefore the spectrum of  $A$  always is of the form

$$(0, \lambda_1, \lambda_2, \dots, \lambda_n, -\lambda_1, -\lambda_2, \dots, -\lambda_n).$$

so  $A$  is traceless  $TrA = 0$ . Taking into account (4) we get  $TrS = 0$ , i.e. each almost Kähler leaf  $N$  is a minimal hypersurface.

Almost cosymplectic manifolds with Kählerian leaves are characterized by the following theorem [9]: an almost cosymplectic manifold  $(M, \varphi, \xi, \eta, g)$  has Kählerian leaves if and only if

$$(\nabla_X\varphi)Y = -g(\varphi AX, Y)\xi + \eta(Y)\varphi AX.$$

For given almost contact structure  $(\varphi, \eta, \xi)$  on a manifold  $M$  we define an almost complex structure  $\tilde{J}$  on  $M \times \mathbb{R}$  as follows

$$\tilde{J}(X, f\frac{d}{dt}) = (\varphi X - f\xi, \eta(X)\frac{d}{dt}).$$

The structure  $(\varphi, \xi, \eta)$  is said to be normal if  $\tilde{J}$  is integrable, i.e. complex structure on  $M \times \mathbb{R}$ .

We say that an almost cosymplectic manifold  $(M, \varphi, \xi, \eta, g)$  is cosymplectic if its almost contact structure  $(\varphi, \xi, \eta)$  is normal. Cosymplectic manifolds are characterized by the condition that the tensor field  $\varphi$  is parallel [1]

$$\nabla\varphi = 0.$$

Thus for cosymplectic manifold we have

$$(6) \quad R(X, Y)\varphi Z = \varphi R(X, Y)Z,$$

that is the collineation  $\varphi$  commutes with the curvature operator. The converse statement is also true [7], [8]: if (6) is satisfied then the manifold  $M$  is cosymplectic.

Now let  $(N, J, G)$  be a  $2n$  dimensional almost Kähler manifold and  $I$  a nonempty open interval. On the product  $N \times I$  we define an almost contact metric structure  $(\varphi, \xi, \eta, g)$  as follows

$$\begin{aligned}\varphi(\bar{X}, f \frac{d}{dt}) &= (J\bar{X}, 0), \quad \xi = \frac{d}{dt}, \quad \eta(\bar{X}, f \frac{d}{dt}) = f, \\ g((\bar{X}, f \frac{d}{dt}), (\bar{X}, f \frac{d}{dt})) &= G(\bar{X}, \bar{X}) + f^2,\end{aligned}$$

here  $f$  denotes a function on  $N \times I$ . It is simply to verify that  $(N \times I, \varphi, \xi, \eta, g)$  is an almost cosymplectic manifold and is cosymplectic if  $N$  is Kähler. We note that for this example we always have  $A = 0$  that is the vector field  $\xi$  is parallel.

From a local point of view an almost cosymplectic manifold  $M$  with vanishing tensor  $A$  has a structure as described in the example above.

### 3. CONFORMALLY FLAT MANIFOLD

A Riemannian manifold  $(M, g)$  is said to be locally conformally flat if any point  $p \in M$  has a neighborhood  $U_p$  and there is a positive function  $f : U_p \rightarrow U_p$  such that a metric  $g' = fg$  is a flat (Euclidean) metric on  $U_p$ .

For a given Riemannian manifold  $(M, g)$  the standard routine to detect the conformal flatness of  $g$  is to verify that some tensor fields determined by the Riemann curvature tensor are vanishing everywhere. If  $\dim M = n \geq 4$  then one should verify that a so called the Weyl curvature tensor vanishes. Precisely let  $R(X, Y)Z$  be the curvature operator  $R(X, Y)Z = [\nabla_X, \nabla_Y]Z - \nabla_{[X, Y]}Z$  and  $S$  a Ricci tensor  $S(X, Y) = \text{Tr} X \mapsto R(X, Y)Z$ . We define a Ricci operator  $Q$  requiring that  $g(QX, Y) = S(X, Y)$  and a scalar curvature  $s = \text{Tr} Q$ . Then the Weyl curvature  $C$  is defined by

$$\begin{aligned}C(X, Y)Z &= R(X, Y)Z - \frac{1}{n-2}(g(Y, Z)QX + g(QY, Z)X \\ &\quad - g(X, Z)QY - g(QX, Z)Y) \\ &\quad + \frac{s}{(n-1)(n-2)}(g(Y, Z)X - g(X, Z)Y).\end{aligned}$$

The Weyl's theorem states that the manifold  $(M, g)$ ,  $\dim M = n \geq 4$  is locally conformally flat if and only if the tensor  $C$  vanishes. The case of the dimension three is different for in this dimension the Weyl curvature vanishes identically. For  $\dim M = 3$  we define a Weyl-Schouten tensor  $L$

$$LX = QX - \frac{s}{4}X.$$

Then  $(M, g)$  is locally conformally flat if and only if

$$(7) \quad (\nabla_X L)Y = (\nabla_Y L)X.$$

#### 4. CONFORMALLY FLAT THREE DIMENSIONAL ALMOST COSYMPLECTIC MANIFOLDS

Let  $(M, \varphi, \xi, \eta, g)$  be a three dimensional almost cosymplectic manifold. Near a point  $p \in M$  we fix an orthonormal frame  $(E_1, E_2, E_3)$  of vector fields

$$\begin{aligned} E_1 &= \xi, & \varphi E_2 &= E_3, & \varphi E_3 &= -E_2, \\ AE_2 &= -\lambda E_2, & AE_3 &= \lambda E_3. \end{aligned}$$

Note that locally such frame always exists. Moreover if  $\lambda \neq 0$  it is determined uniquely up to the change of sign  $(E_1, E_2) \mapsto (-E_1, -E_2)$ . The conditions  $d\eta = d\Phi = 0$  imply that the commutators  $[E_i, E_j]$  should satisfy the following relations [9]

$$(8) \quad \begin{aligned} [E_1, E_2] &= -\lambda E_2 + \alpha E_3, & [E_1, E_3] &= \alpha E_2 - \lambda E_3 \\ [E_2, E_3] &= \beta E_2 - \gamma E_3. \end{aligned}$$

The Jacobi identity yields the following additional conditions

$$(9) \quad \begin{aligned} E_2\lambda - E_3\alpha + E_1\gamma - \alpha\beta + \gamma\lambda &= 0, \\ E_3\lambda - E_2\alpha - E_1\beta - \alpha\gamma + \beta\lambda &= 0. \end{aligned}$$

Note that these system posses an interesting symmetry properties. However the detailed discussion is out of the scope of this paper.

With respect to this frame we obtain the components  $S_{ij}$  of the Ricci tensor [9]

$$\begin{aligned} S_{11} &= -2\lambda^2, & S_{12} &= E_2\lambda + 2\gamma\lambda, & S_{13} &= -(E_3\lambda + 2\beta\lambda), \\ S_{22} &= -E_1\lambda - E_2\gamma - E_3\beta - \beta^2 - \gamma^2, & S_{23} &= -2\alpha\lambda, \\ S_{33} &= E_1\lambda - E_2\gamma - E_3\beta - \beta^2 - \gamma^2. \end{aligned}$$

and the scalar curvature  $s = S_{11} + S_{22} + S_{33}$

$$s = -2E_2\gamma - 2E_3\beta - 2(\beta^2 + \gamma^2 + \lambda^2).$$

Now let assume that  $M \subset \mathbb{R}^3$  is a domain. Let  $p \in M$ ,  $p = (x, y, z)$  and

$$E_1 = \xi = \frac{\partial}{\partial z}, \quad E_2 = (a^1, a^2, a^3), \quad E_3 = (b^1, b^2, b^3),$$

where  $a^i, b^i$  are some functions on  $M$  and

$$(c^1, c^2, c^3) = c^1 \frac{\partial}{\partial x} + c^2 \frac{\partial}{\partial y} + c^3 \frac{\partial}{\partial z}$$

Now (8), (9) and (7) form a nonlinear overdetermined system of differential equations of the second order with respect to the unknown functions  $a^i, b^j, \alpha, \beta, \gamma, \lambda$ .

As we mentioned in the Introduction all cosymplectic locally conformally flat manifolds can be described completely. Indeed we known that a manifold  $M$  of this type is locally a product of an nonempty open interval and the a two dimensional Kähler manifold  $N$ . Therefore  $N$  is of constant sectional curvature.

A non-cosymplectic case is more complicated. Note that the example described in the Introduction can be slightly generalized. Let  $U \subset \mathbb{R}^3$  be a

domain in  $\mathbb{R}^3 = \{(x, y, z) | x, y, z \in \mathbb{R}\}$ . Let  $f = f(x, z) > 0$ ,  $u = u(x) > 0$  be real functions on  $U$ . We define an almost contact metric structure  $(\varphi, \xi, \eta, g)$  as follows

$$\begin{aligned} g &= f(x, z)^2 dx^2 + \frac{u(x)^2}{f(x, z)^2} dy^2 + dz^2, \\ \xi &= \frac{\partial}{\partial z}, \quad \eta = dz, \\ \varphi \frac{\partial}{\partial x} &= \frac{f(x, z)^2}{u(x)} \frac{\partial}{\partial y}, \quad \varphi \frac{\partial}{\partial y} = -\frac{u(x)}{f(x, z)^2} \frac{\partial}{\partial x}. \end{aligned}$$

For the fundamental form we have

$$\Phi = 2u(x)dx \wedge dy.$$

Obviously  $d\eta = d\Phi = 0$ . Rearranging terms we write down  $g$  as follows

$$g = \frac{u(x)^2}{f(x, z)^2} (dy^2 + \frac{f(x, z)^4}{u(x)^2} dx^2 + \frac{f(x, z)^2}{u(x)^2} dz^2).$$

It is evident that  $g$  is conformally flat if and only if the term inside the parenthesis is a conformally flat metric.

**Proposition 1.** *The almost cosymplectic manifold  $(U, \varphi, \xi, \eta, g)$  is conformally flat if and only if the functions  $f$  and  $u$  satisfy the following differential equation*

$$(10) \quad 2\partial_z^2 f - \partial_x^2 \frac{1}{f} - \partial_x \left( \frac{\partial_x \ln u}{f} \right) = -\kappa \frac{f^3}{u^2},$$

for a constant  $\kappa$ .

*Proof.* The metric

$$dy^2 + \frac{f(x, z)^4}{u(x)^2} dx^2 + \frac{f(x, z)^2}{u(x)^2} dz^2,$$

is conformally flat if and only if

$$\frac{f(x, z)^4}{u(x)^2} dx^2 + \frac{f(x, z)^2}{u(x)^2} dz^2,$$

is of constant sectional curvature  $\kappa$ . The latter is equivalent to the functions  $f$  and  $u$  should satisfy (10).  $\square$

We note that the function  $u$  in this equation plays a role of the functional parameter and is not obvious that for a given  $u$  the solution exists. However if  $u$  is real analytic then we can apply the Cauchy-Kovalevska theorem.

**Examples.** Setting  $f(x, z) = \frac{t(z)}{s(x)}$ ,  $t(z) > 0$ ,  $s(x) > 0$  in (10) we obtain

$$2t_z'' - \frac{s((s \cdot u)'_x / u)'_x}{t} = -\kappa \frac{t^3}{(s \cdot u)^2}.$$

If  $\kappa = 0$  then we have two independent equations

$$t_z'' = 0, \quad ((s \cdot u)'_x / u)'_x = 0.$$

Solving these equations one gets

$$(11) \quad g = \frac{u(x)^2(Az + B)^2}{(C \int u(x) + D)^2} dx^2 + \frac{(C \int u(x) + D)^2}{(Az + B)^2} dy^2 + dz^2,$$

as we see  $u$  may be arbitrary function. Note that if  $u = e^{ax}$  then for properly chosen constants  $A, B, C, D$  we obtain the example described in the Introduction.

If  $\kappa \neq 0$  a solution of the form  $t(z)/s(x)$  exists only if  $s \cdot u = C = \text{const} > 0$ . In this case

$$t_z'' = -\frac{\kappa}{2C^2} t^3,$$

which yields

$$z = \pm 2C \int \frac{dt}{\sqrt{D - \kappa t^4}}.$$

The constant  $D$  should be properly chosen depending on  $\kappa$  and a domain of definition of  $t$ . Explicitly the metric is given by

$$(12) \quad g = (Ct)^2 dv^2 + \frac{1}{(Ct)^2} dy^2 + \frac{4C^2}{D - \kappa t^4} dt^2,$$

where  $dv = u(x)dx$ . As above the function  $u$  can be arbitrary.

The examples above have some interesting properties. We note which is evident that the field  $\partial_y$  is Killing. What is more important this is the field of eigenvectors

$$A \frac{\partial}{\partial y} = \partial_z \ln f \frac{\partial}{\partial y}.$$

From this point of view without reference to any particular local chart we may say that these manifolds have the property that there is a smooth field of eigenvectors of the operator  $A$  which is Killing.

**Theorem 1.** *Let  $(M, \varphi, \xi, \eta, g)$  be a three dimensional almost cosymplectic manifold. Assume that a smooth field  $K$  of eigenvectors  $K$  of the operator  $A = -\nabla \xi$ ,  $A \neq 0$  everywhere, is a Killing vector field. Then the manifold  $M$  is conformally flat if and only if the Laplacian  $\Delta \frac{1}{|K|}$  of the inverse of the length of the field  $K$  satisfies the following equation*

$$(13) \quad \Delta \frac{1}{|K|} = \frac{\kappa}{2} \frac{1}{|K|^3},$$

for a constant  $\kappa$ .

*Proof.* Note that  $[\xi, K] = 0$ . Indeed, let  $X$  be arbitrary vector field. Then as  $K$  is Killing and  $A$  symmetric we find

$$(14) \quad 0 = (\mathcal{L}_K g)(\xi, X) = g([\xi, K], X).$$



Now we introduce a local orthonormal frame  $(E_1, E_2, E_3)$  and

$$(15) \quad E_1 = \xi, \quad \varphi E_2 = E_3, \quad E_3 = K/|K|.$$

As  $AE_2 = -\lambda E_2$ ,  $AE_3 = \lambda E_3$  by (15) and (8)

$$[E_1, E_2] = -\xi \ln |K| E_2, \quad [E_1, E_3] = [\xi, E_3] = \xi \ln |K| E_3.$$

As a collorary we get  $\lambda = \xi \ln |K|$ . Therefore all the distributions  $(E_i, E_j)$ ,  $i < j$  are involutive. Equivalently

$$\theta^1 \wedge d\theta^1 = \theta^2 \wedge d\theta^2 = \theta^3 \wedge d\theta^3 = 0$$

where  $\theta^i$  are dual forms. Hence there are locally nonzero functions  $u_i$  such that the frame  $u_i E_i$  is holonomic, i.e  $[u_i E_i, u_j E_j] = 0$ . The functions  $u_i$  are simply “integrability factors”  $d(\theta^i/u_i) = 0$ . It is clear that as  $d\theta^1 = d\eta = 0$  we may assume that  $u_1 = 1$ . We introduce the following denotations

$$\xi = \frac{\partial}{\partial z} = \partial_z, \quad u_2 E_2 = \partial_x, \quad u_3 E_3 = \partial_y.$$

The  $\partial_x, \partial_y, \partial_z$  are eigenvectors fields and as they correspond to different eigenvalues, resp.  $-\lambda, \lambda, 0$  they are pairwise orthogonal. Therefore the metric takes the following form

$$(16) \quad g = f^2 dx^2 + \frac{u^2}{f^2} dy^2 + dz^2,$$

for a functions  $f > 0$ ,  $u > 0$ . It is clear that  $K = \beta \partial_y$  with a nonzero coefficient  $\beta$ . We always can assume that  $\beta = 1$ . Indeed the conditions

$$0 = [\xi, K] = [\partial_z, K], \quad 0 = (\mathcal{L}_K g)(\partial_x, \partial_y),$$

and (16) imply  $K = \beta(y) \partial_y = \partial_y$ . For  $\partial_y$  is Killing and  $d\Phi = 2d(\theta^2 \wedge \theta^3) = 0$  we obtain  $f = f(x, z)$ ,  $u = u(x)$ .

Let  $\Delta$  denote the metric Laplacian, i.e.

$$\Delta v = -Tr\{X \mapsto \nabla_X \text{grad} v\},$$

where  $dv(X) = g(X, \text{grad} v)$ . Taking into account  $u = u(x)$  one verifies that

$$-u \Delta \frac{f}{u} = \partial_z^2 - \partial_x^2 \frac{1}{f} - \partial_x \left( \frac{\partial_x \ln u}{f} \right).$$

Therefore by the above identity the condition (10) can be written as

$$\Delta \frac{f}{u} = \frac{\kappa}{2} \left( \frac{f}{u} \right)^3.$$

Finally  $f/u = 1/|K|$ . □

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